

CENTRED WEIGHTED CONDITIONAL TYPE OPERATORS

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ABSTRACT. In this paper, we give some necessary and sufficient conditions for weighted conditional expectation type operators on $L^2(\Sigma)$ to be centered. Also, we investigate the relation between normal and centered weighted conditional type operators. In the end, we prove that the Aluthage transformation of any weighted conditional expectation type operator is centered, this means a large classes of operators are centered.

1. Introduction and Preliminaries

Let (X, Σ, μ) be a complete σ -finite measure space. For any sub- σ -finite algebra $\mathcal{A} \subseteq \Sigma$, the L^2 -space $L^2(X, \mathcal{A}, \mu|_{\mathcal{A}})$ is abbreviated by $L^2(\mathcal{A})$, and its norm is denoted by $\|\cdot\|_2$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ -null set. The support of a measurable function f is defined as $S(f) = \{x \in X; f(x) \neq 0\}$. We denote the vector space of all equivalence classes of almost everywhere finite valued measurable functions on X by $L^0(\Sigma)$.

For a sub- σ -finite algebra $\mathcal{A} \subseteq \Sigma$, the conditional expectation operator associated with \mathcal{A} is the mapping $f \rightarrow E^{\mathcal{A}}f$, defined for all non-negative, measurable function f as well as for all $f \in L^2(\Sigma)$, where $E^{\mathcal{A}}f$, by the Radon-Nikodym theorem, is the unique \mathcal{A} -measurable function satisfying

$$\int_A f d\mu = \int_A E^{\mathcal{A}}f d\mu, \quad \forall A \in \mathcal{A}.$$

As an operator on $L^2(\Sigma)$, $E^{\mathcal{A}}$ is idempotent and $E^{\mathcal{A}}(L^2(\Sigma)) = L^2(\mathcal{A})$. If there is no possibility of confusion, we write $E(f)$ in place of $E^{\mathcal{A}}(f)$. This operator will play a major role in our work and we list here some of its useful properties:

- If g is \mathcal{A} -measurable, then $E(fg) = E(f)g$.
- $|E(f)|^2 \leq E(|f|^2)$; $f \in L^0(\mathcal{A})$ if and only if $|E(f)|^2 = E(|f|^2)$.
- $|E(fg)| \leq (E(|f|^2))^{\frac{1}{2}}(E(|g|^2))^{\frac{1}{2}}$, (Hölder inequality).
- If $f \geq 0$, then $E(f) \geq 0$; if $f > 0$, then $E(f) > 0$.
- For each $f \geq 0$, $S(f) \subseteq S(E(f))$.

A detailed discussion and verification of most of these properties may be found in [11].

Combinations of conditional expectation operators and multiplication operators appear often in the study of other operators such as multiplication operators and weighted composition operators. Specifically, in [9], S.-T. C. Moy characterized all operators on L^p of the form $f \rightarrow E(fg)$ for g in L^q with $E(|g|)$ bounded. Eleven

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years later, R. G. Douglas, [2], analyzed positive projections on L^1 and many of his characterizations are in terms of combinations of multiplications and conditional expectations. More recently, P.G. Dodds, C.B. Huijsmans and B. De Pagter, [1], extended these characterizations to the setting of function ideals and vector lattices. J. Herron presented some assertions about the operator EM_u on L^p spaces in [6]. Also, some results about multiplication conditional expectation operators can be found in [5, 7]. In [3, 4] we investigated some classic properties of multiplication conditional expectation operators $M_w EM_u$ on L^p spaces. Let $f \in L^0(\Sigma)$, then f is said to be conditionable with respect to E if $f \in \mathcal{D}(E) := \{g \in L^0(\Sigma) : E(|g|) \in L^0(\mathcal{A})\}$. Throughout this paper we take u and w in $\mathcal{D}(E)$.

An operator A on a Hilbert space is *centred* if the family of operators $\{A^{*n} A^n, A^k A^{*k} : n, k \geq 0\}$ is commutative [10].

In this paper we will be concerned with characterizing weighted conditional expectation type operators and their Aluthage transformations on $L^2(\Sigma)$ in terms of membership in the class of centered operators and the relation between normal and centered weighted conditional type operators.

2. Centred and normal weighted conditional type operators

In the first we reminisce some theorems that we have proved in [4].

Theorem 2.1. The operator $T = M_w EM_u$ is bounded on $L^2(\Sigma)$ if and only if $(E|w|^2)^{\frac{1}{2}}(E|u|^2)^{\frac{1}{2}} \in L^\infty(\mathcal{A})$, and in this case its norm is given by $\|T\| = \|(E(|w|^2))^{\frac{1}{2}}(E(|u|^2))^{\frac{1}{2}}\|_\infty$.

Lemma 2.2. Let $T = M_w EM_u$ be a bounded operator on $L^2(\Sigma)$ and let $p \in (0, \infty)$. Then

$$(T^*T)^p = M_{\bar{u}(E(|u|^2))^{p-1}\chi_S(E(|w|^2))^p} EM_u$$

and

$$(TT^*)^p = M_{w(E(|w|^2))^{p-1}\chi_G(E(|u|^2))^p} EM_{\bar{w}},$$

where $S = S(E(|u|^2))$ and $G = S(E(|w|^2))$.

Theorem 2.3. The unique polar decomposition of bounded operator $T = M_w EM_u$ is $U|T|$, where

$$|T|(f) = \left(\frac{E(|w|^2)}{E(|u|^2)} \right)^{\frac{1}{2}} \chi_S \bar{u} E(uf)$$

and

$$U(f) = \left(\frac{\chi_{S \cap G}}{E(|w|^2)E(|u|^2)} \right)^{\frac{1}{2}} w E(uf),$$

for all $f \in L^2(\Sigma)$.

Theorem 2.4. The Aluthge transformation of $T = M_w EM_u$ is

$$\hat{T}(f) = \frac{\chi_S E(uw)}{E(|u|^2)} \bar{u} E(uf), \quad f \in L^2(\Sigma).$$

It is shown in [10] that if A is an operator such that, for each positive n , A^n has polar decomposition $U_n P_n$, then A is centred if and only if for each positive n ,

$U_n = U_1^n$. In the sequel we will characterize centred weighted conditional type operators.

Theorem 2.5. Consider the weighted conditional type operator $M_w E M_u : L^2(\Sigma) \rightarrow L^2(\Sigma)$. Then

- (1) If $M_w E M_u$ is centred, then $|E(uw)|^2 = E(|u|^2)E(|w|^2)$ on $S(E(uw)E(w)E(u))$.
- (2) If $|E(uw)|^2 = E(|u|^2)E(|w|^2)$, then $M_w E M_u$ is centred.

Proof. (a) By induction we have

$$T^n f = (E(uw))^{n-1} w E(uf), \quad f \in L^2(\Sigma), \quad n \in \mathbb{N}.$$

Now, by Theorem 2.3 we obtain

$$U_n(f) = \frac{\chi_H E(uw)^{n-1}}{(E(|u|^2))^{\frac{1}{2}}(E(|w|^2))^{\frac{1}{2}} |E(uw)|^{n-1}} w E(uf), \quad U(f) = \frac{\chi_{S \cap G}}{(E(|u|^2))^{\frac{1}{2}}(E(|w|^2))^{\frac{1}{2}}} w E(uf)$$

and

$$U_1^n(f) = \frac{\chi_{S \cap G} E(uw)^{n-1}}{(E(|u|^2))^{\frac{n}{2}}(E(|w|^2))^{\frac{n}{2}}} w E(uf)$$

for all $f \in L^2(\Sigma)$ and $H = S(E(uw))$. This implies that if $U_n = U_1^n$, then

$$\frac{\chi_H E(uw)^{n-1}}{(E(|u|^2))^{\frac{1}{2}}(E(|w|^2))^{\frac{1}{2}} |E(uw)|^{n-1}} w E(uf) = \frac{\chi_{S \cap G} E(uw)^{n-1}}{(E(|u|^2))^{\frac{n}{2}}(E(|w|^2))^{\frac{n}{2}}} w E(uf)$$

for all $f \in L^2(\Sigma)$. So, for positive element $a \in L^2(\mathcal{A})$,

$$\frac{\chi_H E(uw)^{n-1}}{(E(|u|^2))^{\frac{1}{2}}(E(|w|^2))^{\frac{1}{2}} |E(uw)|^{n-1}} w E(ua) = \frac{\chi_{S \cap G} E(uw)^{n-1}}{(E(|u|^2))^{\frac{n}{2}}(E(|w|^2))^{\frac{n}{2}}} w E(ua)$$

and so

$$\left(\frac{\chi_H E(uw)^{n-1}}{(E(|u|^2))^{\frac{1}{2}}(E(|w|^2))^{\frac{1}{2}} |E(uw)|^{n-1}} - \frac{\chi_{S \cap G} E(uw)^{n-1}}{(E(|u|^2))^{\frac{n}{2}}(E(|w|^2))^{\frac{n}{2}}} \right) w E(u)a = 0.$$

Then

$$|E(uw)|^{n-1} (E(|u|^2))^{\frac{1}{2}} (E(|w|^2))^{\frac{1}{2}} = (E(|u|^2))^{\frac{n}{2}} (E(|w|^2))^{\frac{n}{2}}$$

on $S(E(w)E(u)) \cap H$. Thus

$$|E(uw)|^2 = E(|u|^2)E(|w|^2) \text{ on } S(E(w)E(u)) \cap H.$$

(b) Suppose that $|E(uw)|^2 = E(|u|^2)E(|w|^2)$. By part (a) and direct computation shows that $U_n = U_1^n$. Thus $M_w EM_u$ is centred.

Corollary 2.6. If $S(E(u)E(w)) = S \cap G = H$, then the operator $M_w EM_u$ on $L^2(\Sigma)$ is centred if and only if $|E(uw)|^2 = E(|u|^2)E(|w|^2)$.

Corollary 2.7. Consider the weighted conditional type operator $EM_u : L^2(\Sigma) \rightarrow L^2(\Sigma)$. Then

(1) If EM_u is centred, then $|E(u)|^2 = E(|u|^2)$ on $S(E(u))$.

(2) If $|E(u)|^2 = E(|u|^2)$, then EM_u is centred.

Corollary 2.8. If $S(E(u)) = S(E(|u|^2))$, then the operator EM_u on $L^2(\Sigma)$ is centred if and only if $u \in L^0(\mathcal{A})$.

Recall that each operator A on a Hilbert space \mathcal{H} is called normal if $A^*A = AA^*$. In the sequel some necessary and sufficient conditions for normality will be presented.

Theorem 2.9. Let $T = M_w EM_u$ be a bounded operator on $L^2(\Sigma)$, then

- (a) If $(E(|u|^2))^{\frac{1}{2}}\bar{w} = u(E(|w|^2))^{\frac{1}{2}}$, then T is normal.
- (b) If T is normal, then $|E(u)|^2 E(|w|^2) = |E(w)|^2 E(|u|^2)$.

Proof. (a) Applying lemma 2.2 we have

$$T^*T - TT^* = M_{\bar{u}E(|w|^2)}EM_u - M_{wE(|u|^2)}EM_{\bar{w}}.$$

So for every $f \in L^2(\Sigma)$,

$$\begin{aligned} \langle T^*T - TT^*(f), f \rangle &= \\ &= \int_X E(|w|^2)E(uf)\bar{u}f - E(|u|^2)E(\bar{w}f)w\bar{f}d\mu \\ &= \int_X |E(u(E(|w|^2))^{\frac{1}{2}}f)|^2 - |E((E(|u|^2))^{\frac{1}{2}}\bar{w}f)|^2 d\mu. \end{aligned}$$

This implies that if

$$(E(|u|^2))^{\frac{1}{2}}\bar{w} = u(E(|w|^2))^{\frac{1}{2}},$$

then for all $f \in L^2(\Sigma)$, $\langle T^*T - TT^*(f), f \rangle = 0$, thus $T^*T = TT^*$.

(b) Suppose that T is normal. By (a), for all $f \in L^2(\Sigma)$ we have

$$\int_X |E(u(E(|w|^2))^{\frac{1}{2}}f)|^2 - |E((E(|u|^2))^{\frac{1}{2}}\bar{w}f)|^2 d\mu = 0.$$

Let $A \in \mathcal{A}$, with $0 < \mu(A) < \infty$. By replacing f to χ_A , we have

$$\int_A |E(u(E(|w|^2))^{\frac{1}{2}})|^2 - |E((E(|u|^2))^{\frac{1}{2}}\bar{w})|^2 d\mu = 0$$

and so

$$\int_A |E(u)|^2 E(|w|^2) - |E(w)|^2 E(|u|^2) d\mu = 0.$$

Since $A \in \mathcal{A}$ is arbitrary, then $|E(u)|^2 E(|w|^2) = |E(w)|^2 E(|u|^2)$. \square

Corollary 2.10. The operator EM_u on $L^2(\Sigma)$ is normal if and only if $u \in L^\infty(\mathcal{A})$.

Corollary 2.11. Consider the weighted conditional type operator $T = EM_u : L^2(\Sigma) \rightarrow L^2(\Sigma)$. If $S(E(u)) = S(E(|u|^2))$, then the following conditions are equivalent:

- (1) T is centered.
- (2) T is normal.
- (3) $u \in L^\infty(\mathcal{A})$.

The next theorem state that a large classes of operators are centered. That will be a nice result, because of importance of centered operators.

Theorem 2.12. The Aluthge transformation \widehat{T} of the operator $T = M_w EM_u$ is always centered.

Proof. By Theorem 2.4 we have

$$\widehat{T}(f) = \frac{\chi_S E(uw)}{E(|u|^2)} \bar{u} E(uf), \quad f \in L^2(\Sigma).$$

This means that $\widehat{T} = M_{w'} EM_u$, where $w' = \frac{\chi_S E(uw) \bar{u}}{E(|u|^2)}$. So we have

$$|E(uw')|^2 = |E(u \frac{\chi_S E(uw) \bar{u}}{E(|u|^2)})|^2 = (E(|u|^2))^2 \frac{\chi_S |E(uw)|^2}{(E(|u|^2))^2} = |E(uw)|^2.$$

Also,

$$E(|u|^2) E(|w'|^2) = E(|u|^2) \frac{\chi_S E(|u|^2) |E(uw)|^2}{(E(|u|^2))^2} = |E(uw)|^2,$$

This implies that $|E(uw')|^2 = E(|u|^2) E(|w'|^2)$, so by Theorem 2.5 the Aluthge transformation \widehat{T} of the operator $T = M_w EM_u$ is centered.

Example 2.13. Let $X = [0, 1] \times [0, 1]$, $d\mu = dx dy$, Σ the Lebesgue subsets of X and let $\mathcal{A} = \{A \times [0, 1] : A \text{ is a Lebesgue set in } [0, 1]\}$. Then, for each f in $L^2(\Sigma)$, $(Ef)(x, y) = \int_0^1 f(x, t) dt$, which is independent of the second coordinate. This example is due to A. Lambert and B. Weinstock [8]. Now, if we take $u(x, y) = y^{\frac{x}{2}}$ and $w(x, y) = \sqrt{(4+x)y}$, then $E(|u|^2)(x, y) = \frac{4}{4+x}$ and $E(|w|^2)(x, y) = \frac{4+x}{2}$. So, $E(|u|^2)(x, y) E(|w|^2)(x, y) = 2$ and $|E(uw)|^2(x, y) = 64 \frac{4+x}{(x+12)^2}$. Direct computations shows that

$$E(|u|^2)(x, y) E(|w|^2)(x, y) \leq |E(uw)|^2(x, y).$$

Since $T = M_w EM_u$ is bounded then we have

$$E(|u|^2)(x, y) E(|w|^2)(x, y) \geq |E(uw)|^2(x, y),$$

this implies that $E(|u|^2)(x, y)E(|w|^2)(x, y) = |E(uw)|^2(x, y)$. So by Theorem 2.5 the operator $T = M_wEM_u$ is centered.

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